

# Groupoids & $C^*$ -algebras

Recall notion of category  $\mathcal{C}$ :

- $\mathcal{C}_0 := \text{Object}(\mathcal{C}) = \text{Set of objects}$
- $\mathcal{C}_1 := \text{Hom}_{\mathcal{C}}(\mathcal{C}) = \text{Set of homs.}$
- $d_0: \mathcal{C}_1 \rightarrow \mathcal{C}_0$  domain / Sources  $d_0(a \xleftarrow{b}) = b$
- $c_0: \mathcal{C}_1 \rightarrow \mathcal{C}_0$  codomain / target  $c_0(a \xleftarrow{b}) = a$
- $\mathcal{C}_2 := \text{Set of Composable morphism} := \text{pullback of}$   $\begin{array}{ccc} \mathcal{C}_2 & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow c_0 \\ \mathcal{C}_1 & \xrightarrow{d_0} & \mathcal{C}_0 \end{array}$
- (Pull back:  $\begin{array}{ccc} X & & \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} := \{(z, x) : g(z) = f(x)\} \\ \downarrow f & & \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$ )
- $\square: \mathcal{C}_2 \rightarrow \mathcal{C}_1$   $\quad 1, (f \square g) \square h = f \square (g \square h)$
- $\square(a \xleftarrow{b} c) = a \xleftarrow{c}$  such that:
  - 2,  $d_0 \square u = \text{id}_{\mathcal{C}_1}$  &  $u \square \text{id} = \text{id}_{\mathcal{C}_2}$  in which  $u: \mathcal{C}_0 \rightarrow \mathcal{C}_1$  is unit
  - 3,  $f \square u \circ v = f = u \circ b \square f$  if  $d_0(f) = a$  and  $c_0(f) = b$ .

Notations:  $\text{Hom}_{\mathcal{C}}(a, b) := \text{pullback of the following diagram:}$

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(a, b) & \dashrightarrow & \mathcal{C}_1 \\ \downarrow & \text{id}_{\mathcal{C}_0} & \downarrow d_0 \times_{\mathcal{C}_0} \\ \{a, b\} & \hookrightarrow & \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \end{array} \quad \text{Hom}_{\mathcal{C}_1} := \bigsqcup_{a, b \in \mathcal{C}_0} (\text{Hom}_{\mathcal{C}}(a, b))$$

Examples:

- 1, Category of sets:  $\mathcal{C}_0 = \{\text{sets}\}$   $\mathcal{C}_1 = \{\text{maps between sets}\}$   $\square = \text{ordinary composition}$

2, Category of Hilbert space:

$$\mathcal{C}_0 = \{\text{Hilbert spaces}\} \quad \mathcal{C}_1 = \{\text{unitary maps}\}$$

3) Category of Continuous maps  $C_0 : \begin{matrix} E \\ \downarrow f \\ B \end{matrix} \quad C_1 \quad \begin{matrix} E' \\ \downarrow f' \\ B' \end{matrix}$

4) Find a top space  $B$  define  $\begin{matrix} B \\ \downarrow f \\ B \end{matrix}$   $obj(\begin{matrix} B \\ \downarrow f \\ B \end{matrix}) = B$ ,  $Hom(\begin{matrix} B \\ \downarrow f \\ B \end{matrix}) = B$

$$d_0 = c_0 = u = id, \quad \square(\begin{matrix} B \\ \downarrow f \\ B \end{matrix}) = \frac{B \times B}{B} \cong B$$

5) G a group, define  $\begin{matrix} G \\ \downarrow f \\ * \end{matrix}$

$$obj(\begin{matrix} G \\ \downarrow f \\ * \end{matrix}) = \{*\}, \quad Hom(\begin{matrix} G \\ \downarrow f \\ * \end{matrix}) = G$$

$$d_0 = c_0 = \text{constant} \quad \square := \text{group multiplication}$$

6)  $X$  = space,  $G$  = group  $\therefore X \times G \xrightarrow{\text{action}} X$ , define  $\begin{matrix} X \times G \\ \downarrow f \\ X \end{matrix}$

$$obj(\begin{matrix} X \times G \\ \downarrow f \\ X \end{matrix}) = X, \quad Hom(\begin{matrix} X \times G \\ \downarrow f \\ X \end{matrix}) = X \times G$$

$$\begin{matrix} X & \xleftarrow{(X,g)} & X \cdot g & \xleftarrow{(X \cdot g, h)} & X \cdot g \cdot h \\ (X,g) & & & (X \cdot g, h) & \end{matrix}$$

7: Find a top space  $B$ :

$$obj(\begin{matrix} E \\ \downarrow f \\ B \end{matrix}), \alpha \in H^3(E, \mathbb{Z}) \quad mor(\begin{matrix} E \\ \downarrow f \\ B \end{matrix}, \alpha) \rightarrow (\begin{matrix} E' \\ \downarrow f' \\ B \end{matrix}, \alpha')$$

$$\text{such that } \begin{matrix} E & \xrightarrow{f} & E' \\ p \searrow & \nwarrow & p' \\ B & / & B' \end{matrix}, \quad \alpha = f^* \alpha' + p^* \alpha'$$

Def: A groupoid is a category where all morphisms are isomorphisms

Def: A topological groupoid is a groupoid such that all structures maps are continuous between 1. cpt +  $T_2$  + 2nd cont. top spaces  $C_1, C_0$ .

Example 4, 5, 6 are top. groupoids.

7,

Def. Let  $G_1 \rightrightarrows G_0$  be a top. groupoid. A Haar system on

$G_1 \rightrightarrows G_0$  is a family of (Borel) measures  $\{\lambda^a\}_{a \in G_0}$  on  $E$

$a)$  such that  $\text{Supp } \lambda^a \subset d^{-1}(\{a\}) \subset G_1$

$$\begin{aligned} \int_{d^{-1}(b)} F(x) d\lambda^a(x) &= \int_{d^{-1}(b)} F(x) d\gamma^*(\gamma^a(x)) \\ &\stackrel{(b)}{=} \int_{d^{-1}(b)} F(x) d\lambda^a(x) \end{aligned}$$

(in term of integral)

$b)$   $(\gamma^*)^* \lambda^a = \lambda^b$  where  $\gamma: b \rightarrow a$

$c)$  if  $f \in C_c(G_1)$ , then  $\int_{G_1} f d\lambda^a \in \mathbb{C}$  is continuous

" (X,  $\mu$ ) = Borel measurable space, (a)  $\text{Supp } \mu = X \setminus \bigcup_n U_n$ , (b)  $\varphi: X \rightarrow Y$  measurable map

$$\begin{array}{c} u \subseteq X \\ \text{open} \\ \mu(u) = 0 \end{array}$$

$\varphi_* \mu = \text{Push forward of } \mu$   
along  $\varphi$  by

$$\varphi_* \mu(A) := \mu(\varphi^{-1}(A)).$$

Construct Haar systems in examples:

$$G_1 \quad \begin{array}{c} \uparrow \downarrow \\ B \\ \downarrow \end{array} \quad d^{-1}(\{a\}) = \{a\} \quad \lambda^a(\{a\}) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases} \quad A \subset G_1$$

5) Special Case of example 6.

$$\begin{array}{ccc} & (xg, g) & (x, h) \\ 6) \quad X \times G & x \cdot \tilde{g}^{-1} \leftarrow x \rightarrow x \cdot h & \end{array}$$

- $d^{-1}(X) = \{(x \cdot \tilde{g}^{-1}, g) : g \in G\} \rightleftharpoons G$
- Let  $\lambda$  be a Haar meas. of  $G$  then  $(h)_! \lambda = \lambda$  in which  $h^*: G \rightarrow G$   
 $g \mapsto g \cdot h$

• Set  $\lambda^x = \varphi_{x!} \lambda$

$$(x, h)_! \lambda^x = (x, h)_! \varphi_{x!} \lambda = ((x, h)_! \varphi_x)_! \lambda = (\varphi_{xh} \circ h)_! \lambda = \varphi_{xh!}^{*} h_! \lambda$$
$$\begin{array}{c} \varphi_x \\ \downarrow \end{array} \quad \begin{array}{c} (x, g \circ g) \\ \longrightarrow (x_g^{-1}, g, h) \end{array}$$
$$\begin{array}{c} \varphi_h \\ \downarrow \end{array} \quad \begin{array}{c} g_h \\ \longrightarrow (xh(g_h)^{-1}, g_h) \end{array} =$$
$$= \varphi_{xh!}^{*} \lambda = \lambda^{xh}$$

## Ansgar (II)

We introduce the top groupoid

$$\begin{array}{ccccc} B & \times & C & \times C & B \times B \\ \downarrow & & \downarrow & & \downarrow \\ B & * & X & & B \end{array}$$

• Haar system: family of borel

$$\text{measure } \{\delta_a^x\}_{a \in G_0}, \quad \begin{matrix} G \\ \text{top gr} \\ G_0 \end{matrix}$$

$$\text{on supp } \delta^a = \text{do}^{-1}(a)$$

b) invariance

c) continuity

$$\text{Ex: } B \times B \quad \text{do}^{-1}(b) = \{(a, b) \in B \times B\}$$

$$\begin{matrix} & & f_b \\ & b^a & \\ \downarrow & & \uparrow \\ B & & \end{matrix} \quad \begin{matrix} (a, b) \\ \uparrow \\ B \end{matrix}$$

choose a measure  $\mu$  on  $B$  with  $\text{supp } \mu = B$

$$x^b = q_{b!} \cdot \mu \quad (b, c) * \mu = x^c$$

$$(b, c) * q_{b!} \mu = ((b, c)^* q_b!) \cdot \mu =$$

$$q_c! \cdot \mu = x^c$$

This example shows that the existence  
of Haar system is not unique.

Let  $\mu$  be a top groupoid.  
 $\downarrow$   
 $G_0$

Fix  $(f^a)_{a \in G_0}$  some Haar  
system.

Consider  $C_c(G_0)$  and  
define

$$* : C_c(G_0) \times C_c(G_0) \rightarrow C_c(G_0)$$

$f \mapsto (g \mapsto f(g^{-1}))$

$$\text{check } f^{**} = f$$

$$* : C_c(G_0) \times C_c(G_0) \rightarrow C_c(G_0)$$

$$(f, f') \mapsto f * f'$$

$$f * f'(\gamma) := \int F(\gamma s^{-1}) f'(s) \text{ do}(s) (\gamma)$$

claim:  $*$  is associative  
 $\downarrow$   
 (convolution)

Proof:

$$\begin{aligned} (f * f') * f''(\gamma) &= \\ &\int (f * f')(\gamma s^{-1}) f''(s) \text{ do}(s) (\gamma) = \\ &\int \int f(\gamma s^{-1} \epsilon) f'(\epsilon^{-1}) f''(s) \text{ do}(B) \text{ do}(s) (\gamma) = \\ &= \dots \end{aligned}$$

$$\underline{\text{check}} \quad (f * f')^* = f'^* * f^*$$

i.e.

$(C_c(\mathbb{C}), *, +)$  is an  
involutive algebra.

define a norm

$$\|f\|_{\infty, 1} = \sup_{\max} \{ \sup_{a \in \mathbb{Q}_0} \|f\|_{L^1(C_c, \mathbb{F})} : a \in \mathbb{Q}_0 \} \sup_{a \in \mathbb{Q}_0} \|f^*\|_{L^1(C_c, \mathbb{F})}$$

$$\cdot \text{Get a Banach } *-\text{alg } \overline{C_c(\mathbb{Q}_1)}$$

LEMMA: This Banach  $*-\text{alg}$  has a (b) approximate identity.

(This lemma enables us to define enveloping  $C^*\text{-alg}$ )

$* C^*(\overline{C_c(\mathbb{Q}_1)})$  = enveloping  $C^*\text{-alg}$  of  $\overline{C_c(\mathbb{Q}_1)}$ .

$$\underline{\text{claim}}: C^*\left(\begin{smallmatrix} B \\ \downarrow b \\ B \end{smallmatrix}\right) = C_c(B)$$

$$\underline{\text{Pf}}: (f * f')(a) = \int f(a z b^{-1}) f'(b) dz \stackrel{\text{def. of }}{=} (b) = \\ = \int f(a z a^{-1}) f'(z) dz \stackrel{\text{def. of }}{=} (a) = \\ f(a) \cdot f'(a)$$

$*$  = complex convolution

$$f^* = \bar{f}$$

$$\|\cdot\|_{\infty, 1} = \|\cdot\|_{\sup}$$

$$\underline{\text{claim}} \quad C^*(\underset{*}{\frac{G}{\downarrow \downarrow}}) = C^*_{\ell}(B),$$

$$\underline{\text{claim}} \quad C^*(\underset{x}{\frac{G \times X}{\downarrow \downarrow}}) = C_0(X) \times G,$$

$$\underline{\text{claim}} \quad C^*(\underset{B}{\frac{B \times B}{\downarrow \downarrow}}) = k(L^2(B, M))$$

Proof,

$$\begin{array}{c} (a,b) \\ \curvearrowleft b \\ a \end{array} \begin{array}{c} (b,c) \\ \curvearrowright c \\ (a,b) \end{array}$$

$$\begin{aligned} (f + f')(a, b) &= \int f((a, b) \circ (c, b)^{-1}) f'((c, b) \circ a(b)) d\mu(b) \quad q_b(c) = (c, b) \\ &= \int f((a, b) \circ q_b(c)^{-1}) f'(q_b(c)) d\mu(c) \\ &= \int f(a, c) f'(c, b) d\mu(c) \end{aligned}$$

Hilbert-Schmidt operators

$$k_F : L^2(B, M) \rightarrow L^2(B, M)$$

$$k_f(F) = \int f(\cdot, b) F(b) d\mu(b)$$

$$\text{HSop}^{\text{op}} = k_0(L^2(B, M))$$

in

\* \* \* \* \*

Goal: construct a 2-functor

$$\text{Grpd} \longrightarrow \text{CAlg}$$

↑  
2-groupoid of  
top groupoid

↑  
2-groupoid of  
C-algebra

DEF Set  $Z$  be a space. An action of top groupoid  $\frac{G}{\downarrow \downarrow}$  consists of

$$a: Z \xrightarrow{q} G_0$$

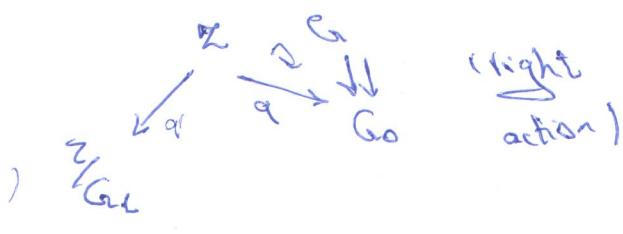
$$b_1: Z \times G_1 \xrightarrow{q_{1,0}} Z$$

$$c: q(Z \times G_1) = da(q)$$

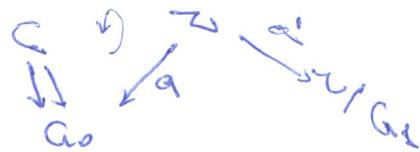
s.t

$$(Z \cdot g) \cdot h = Z \cdot (g \cdot h) \text{ if defined}$$

(quotient by the action)



\* Analogously left actions

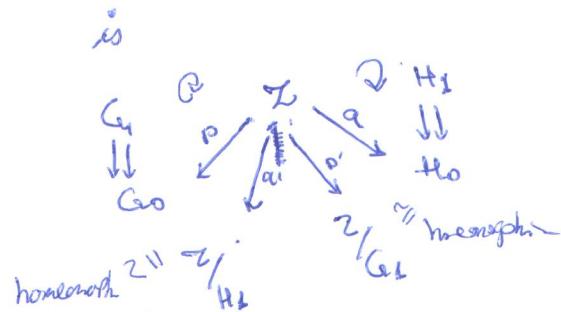


The action  $G$  is called

- free if  $Z \cdot g = Z \Rightarrow g$  is a unit
- proper if  $Z \times G \rightarrow Z \times Z$   
 $\downarrow_{\text{isom}}$   
 $(Z, g) \longmapsto (Z, Z \cdot g)$

is "proper map".

DEF: A Morita equivalence between two top graphs  $\frac{G_1}{\downarrow_{a_1}} \text{ and } \frac{H_1}{\downarrow_{a_1}}$  and  $\frac{G_0}{\downarrow_{a_0}} \text{ and } \frac{H_0}{\downarrow_{a_0}}$



s.t  
two actions commute

- are free
- are proper

— : Think of this as  $Z : \frac{G_1}{\downarrow_{a_1}} \rightarrow \frac{H_1}{\downarrow_{a_1}}$

Let  $\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ \downarrow & & \downarrow \\ G_0 & \xrightarrow{F_0} & H_0 \end{array}$  be a JS factor, then define

$$Z_F := \frac{G_0 \times H_1}{F_0, G_0} = \{(a, \gamma) : a \in Z \xrightarrow[F_0]{\gamma} \}$$

$Z_F$  has a right  $H_1$ -action

$$\begin{array}{ccc} Z_F & \xrightarrow{\alpha} & H_1 \\ & (\alpha, \gamma) \mapsto \alpha(\gamma) & \end{array}$$

$$\begin{array}{ccc} Z_F \times H_1 & \xrightarrow{\alpha \circ \gamma} & Z_F \\ \alpha, \gamma & & \\ ((\alpha, \gamma), \delta) & \mapsto & (\alpha, \gamma \circ \delta) \end{array}$$

$$\frac{Z_F}{H_1} \cong G_0$$

$Z_F$  has canonical left action by  $G_1$

$$\begin{array}{ccc} Z_F & \xrightarrow{\beta} & G_0 \\ & & \\ & (\alpha, \gamma) \mapsto \alpha & \end{array}$$

$$\begin{array}{ccc} G_1 \times Z_F & \rightarrow & Z_F \\ \text{defn} & & \\ (\alpha, (\beta, \gamma)) & \mapsto & (\beta, (\alpha \circ \gamma)) \end{array}$$

Demand: Think of  $Z_F$  as a morphism  $Z_F : \begin{array}{ccc} G_1 & \xrightarrow{\beta} & H_1 \\ \downarrow & & \downarrow \\ G_0 & \xrightarrow{F_0} & H_0 \end{array}$

If  $(F_1, F_2)$  is an isomorphism, then  $Z_F$  is Morita equivalent.

Thm

$$\text{Pf: } G_0 \times H_2 \xrightarrow[F_1, F_2]{\psi} G_2 \times H_0$$

Note that  $\psi$  is bi-equivalent for the canonical actions.

$$\begin{array}{ccc} & \gamma & \\ & \downarrow & \\ a \mapsto x & \longmapsto & \int_{F_1(\gamma)}^{F_2(\gamma)} y \\ F_0 & & \end{array}$$

# Angar (III)

Recall:

$$\begin{array}{ccc} G_1 & \xrightarrow{\quad Z \quad} & H_1 \\ \Downarrow & & \Downarrow \\ G_0 & \xrightarrow{\quad Z \quad} & H_0 \end{array}$$

Morita equivalences are called 1-morphisms

$$C\left(\begin{smallmatrix} X/G_1 \\ \Downarrow \\ X/G_0 \end{smallmatrix}\right) \xrightarrow{\cong} C(X/G_1) \xrightarrow{\text{Morita}} C\left(\begin{smallmatrix} X \times G \\ \Downarrow \\ X \end{smallmatrix}\right) \xrightarrow{\cong} C(X) \times G$$

How to compose 1-morphisms?

Example: Let  $X \times G \rightarrow X$  free and proper action

$$\begin{array}{ccc} X/G & \xrightarrow{\quad Z \quad} & X \times G \\ \Downarrow & \swarrow \text{id} & \Downarrow \\ X/G_1 & & X \\ & \xrightarrow{\quad (X, g) \quad} & \end{array}$$

Non-example: Consider  $S^1 \times \mathbb{Z} \rightarrow S^1$ ,  $\theta \in \mathbb{R} \setminus \mathbb{Q}$

$$(z, n) \mapsto z e^{2\pi i n \theta}$$

This action is free but not proper.

The quotient  $S^1/\mathbb{Z}$  has trivial topology:

$$\tau = \{\emptyset, S^1/\mathbb{Z}\}.$$

$$\text{So } C(S^1/\mathbb{Z}) \cong \mathbb{C}$$

Starting Point of non-comm. Topology

Category of cpt +  $T_2$  Space  $\xrightarrow{\sim} (\text{unital } C^*\text{-alg})^\text{op}$

$$\subseteq (\text{unital } C^*\text{-alg})^\text{op}$$

$$C\left(\begin{smallmatrix} S^1 \times \mathbb{Z} \\ \Downarrow \\ S^1 \end{smallmatrix}\right) \in \text{Cat of non-comm. cpt top Space}$$

$$\begin{array}{ccccc} G_1 & \xrightarrow{\quad Y \quad} & H_1 & \xrightarrow{\quad Z \quad} & K_1 \\ \Downarrow & \nearrow p & \Downarrow & \nearrow q & \Downarrow \\ G_0 & & H_0 & & K_0 \\ & \xrightarrow{\quad r \quad} & & \xrightarrow{\quad s \quad} & \end{array}$$

$$(Y, Z) \circ (X, Y) = (X, Y, Z^{-1}, X, Y)$$

$$\text{Define } Y \square Z := Y \times Z /_{H_0 \cap H_1}$$

$$\begin{array}{ccccc} & [x, y, z] & & [y, z] & [y, z, \delta] \\ \nearrow r & & \searrow p_{(y)} & \nearrow q_{(z)} & \searrow \delta \\ & & & & \end{array}$$

These actions commute and proper.

What about associativity?

$$(Y \square Z) \square W \neq Y \square (Z \square W) \cong (\text{biequivariant})$$

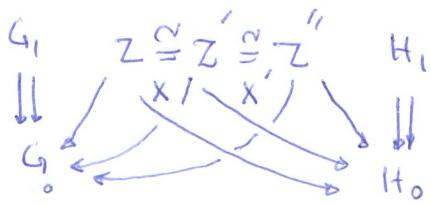
$$[[y, z], w]_K \longrightarrow [y, [z, w]]_{K \cap H}$$

Def: A 2-morphism is bi-equivariant homeomorphism:

$$\begin{array}{ccc} G_1 & \xrightarrow{\quad Z' \quad} & H_1 \\ \Downarrow & \nearrow Z & \Downarrow \\ G_0 & & H_0 \end{array}$$

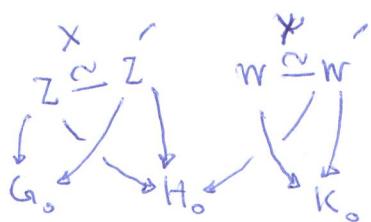
Observe : • 2-morphisms can be composed

vertically



• 2-morphisms can be composed

Horizontally

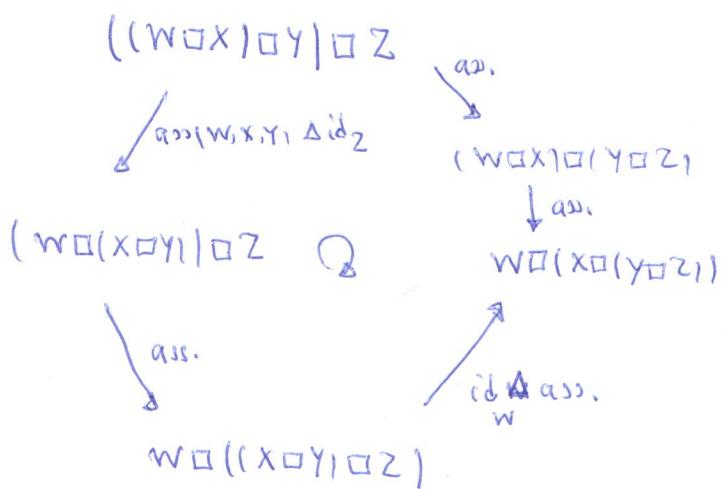


$$x \Delta t \circ z \square w \cong z' \square w'$$

~~•  $(x \circ x) \Delta (\psi \circ \psi) = (x \Delta \psi) \circ (x \Delta \psi)$~~

Note: associativity satisfies the

Pentagon relation



What about units?

$$\begin{array}{ccccccc} G_1 & G_1 & G_1 & Z & H_1 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ G_0 & G_0 & G_0 & H_0 & H_0 \end{array}$$

$$\ell: G_1 \square Z \neq Z \quad \cong$$

$$r: Y \square G_1 \neq Y \quad \cong$$

$$\begin{array}{ccc} G_1 \square G_1 & \xrightarrow{\ell} & G_1 \\ r \downarrow & \nearrow id & \\ G_1 & & \end{array}$$

Def: A 2-groupoid (weak 2 groupoid)

consists of • set of objects

• " " 1-morphisms

• " " 2-morphisms

•  $\square, \circ, \Delta, \ell, r, i, ass, \dots$

such that various coherence conditions are satisfied.

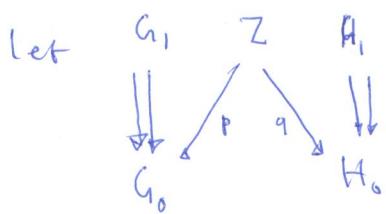
Observation:  $c^*$ -alg form a 2-groupoid

•  $ab = c^*$ -alg

• 1-morphism = Morita equivalence

• 2-morphism = Iso. of Morita equiv.

Ansgar (IV)



be a morita equivalence.

Let  $\{\lambda^a\}_{a \in G_1}$  and  $\{\rho^a\}_{a \in H_0}$

Haar systems on  $G_0, H_0$ .

Take  $\varphi, \psi \in C_c(Z)$ , for  $f \in C_c(G_1)$   
 $g \in C_c(H_1)$

we define for any  $z \in Z$

$$i) (\varphi * g)(z) = \int f(\varepsilon^{-1}) \varphi(\varepsilon.z) d\lambda_{\varepsilon}^a$$

$$ii) (\varphi * g)(z) = \int \varphi(z, \tau) g(\tau) d\rho_{\tau}^a$$

$$iii) \langle \varphi, \psi \rangle_H = \int \varphi(\varepsilon.z) \overline{\psi(\varepsilon.z)} d\mu_{\varepsilon}^a$$

$$iv) \langle \varphi, \psi \rangle_{H_0} = \int \varphi(xz\eta^{-1}) \overline{\psi(z\eta)} d\rho_{(x\eta)}^a$$

Claim: The functions defined in (i)-(iv)

have cpt support, and

$$a) f \cdot (f \cdot \varphi) = (f * f) \circ \varphi$$

$$b) (\varphi * g) * h = \varphi * (g * h)$$

$$c) \langle \varphi, \psi \cdot g \rangle_H = \langle \varphi, \psi \rangle_H \cdot g$$

$$d) \langle f \cdot \varphi, \psi \rangle = f \cdot \langle \varphi, \psi \rangle$$

$$e) \langle \varphi, \psi \rangle^* = \langle \varphi_2, \psi_1 \rangle_H$$

$$f) \langle \varphi_1, \varphi_2 \rangle^* = \langle \varphi_2, \varphi_1 \rangle$$

$$g) \langle \varphi, g, \psi \rangle = \langle \varphi, \psi, g^* \rangle$$

$$h) \langle \varphi, f, \psi \rangle = \langle f \cdot \varphi, \psi \rangle_H$$

$$i) \langle \varphi, \psi \rangle \cdot X = \varphi \cdot \langle \psi, X \rangle_H$$

Check: Density and continuity

$$\| \varphi \|_Z := \| \langle \varphi, \varphi \rangle \|_H^{1/2} = \| \langle \varphi, \varphi \rangle_H \|_H^{1/2}$$

$$C_c(G_1) \xrightarrow{*} C_c(H_2)$$

$$C_c(H_1) \xrightarrow{*} C_c(H_0)$$

Conclusion: Getting a Morita equivalence

$$\frac{U \cdot U_2}{C_c(Z)}$$

$$C_c(G_1) \xrightarrow{*} C_c(H_1)$$

$$C_c(H_0) \xrightarrow{*} C_c(H_1)$$

start with

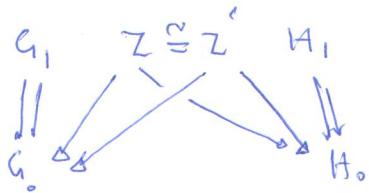
$$\begin{array}{ccccc} G_1 & Y & H_1 & Z & K_1 \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ G_0 & & H_0 & & K_0 \end{array}$$

Then

$$\begin{array}{ccc} \frac{U \cdot U_Y}{C_c(Y)} & & \frac{U \cdot U_Z}{C_c(Z)} \\ C_c(G_1) \xrightarrow{*} & C_c(H_1) \xrightarrow{*} & C_c(K_1) \xrightarrow{*} \\ C_c(G_0) & C_c(H_0) & C_c(K_0) \end{array}$$

$$\begin{array}{ccc} \frac{U \cdot U}{C_c(X)} & \otimes & \frac{U \cdot U}{C_c(Z)} \\ C_c(Y \square Z) & \curvearrowright & C_c(H_1) \xrightarrow{*} \\ C_c(Y \square Z) & & C_c(H_0) \end{array}$$

Start with:



Then the assignment

$Bh: \text{Top}(X) \rightarrow \text{Set theoretical groupoids}$

$$U \mapsto BG_U(U)$$

is a top. stack.

Then we get

$$\begin{array}{ccc} U, h & \xrightarrow{\cong} & U, h \\ C_c(Z) & =^N & C_c(Z') \\ \text{equivariant} & & \end{array}$$

In fact

Groupoid  $\xrightarrow{N}$  top stacks over  $X$

$$\begin{array}{ccc} G_1 & & \\ \downarrow & \nearrow & \\ G_0 & \xrightarrow{\quad} & BG_1(\cdot) \end{array}$$

Conclusion: We have a 2-functor:

$$\text{Groupoid} \xrightarrow{*} \text{C-alg}$$

Top. groupoids are a starting point for  
the study of top. stacks:

Fix  $X$  a space. Consider the category

of principal  $(\begin{smallmatrix} G_1 \\ \downarrow \\ G_0 \end{smallmatrix})$ -bundles

$$\begin{array}{ccc} \text{obj} & E & \xrightarrow{N} EXh \xrightarrow{\cong} EXE \\ \text{has local} & \text{p} \downarrow & \text{G}_1 \downarrow \quad \text{G}_0 \downarrow \\ \text{sections} & X & \text{G}_0 \\ \text{mor:} & \begin{array}{c} \text{equivariant} \\ E \xrightarrow{*} E' \\ \downarrow \quad \downarrow \\ X \end{array} & \end{array}$$

This category is groupoid and its  
denoted by  $BG(X)$ .

Let  $\text{Top}(X)$  be the category of  
open sets of

$$\text{obj} = \{U \subseteq X : \text{open}\} \quad \text{mor} = \left\{ \begin{array}{c} U \subseteq U' \\ \uparrow p \quad \downarrow q \\ X \end{array} \right\}$$