Groupids & C^*-algebras

Recall notion of Category $\mathcal{C}$;

- $\mathcal{C}_0 :=$ Object$(\mathcal{C}) =$ Set of objects $\cdot$
- $\mathcal{C}_1 :=$ Hom$(\mathcal{C}) =$ Set of homs. $\cdot \xleftarrow{f} \cdot$
- $\mathrm{do} : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ domain / Sources $\mathrm{do}(a \leftarrow b) = b$
- $\mathrm{Co} : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ Codomain / target $\mathrm{co}(a \leftarrow b) = a$
- $\mathcal{C}_2 :=$ Set of Composable morphism $:=$ pullback of

$$\begin{tikzcd} \mathcal{C}_2 \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "\mathrm{Co}"] \\ \mathcal{C}_1 \arrow[r, "\mathrm{do}"'] & \mathcal{C}_0 \end{tikzcd}$$

(Pull back :

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

$Z \times_Y X := \{(z, x) : g(z) = f(x)\}$)

- $\square : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$

$\square(a \leftarrow b \leftarrow c) = a \leftarrow c$ such that :

1, $(f \square g) \square h = f \square (g \square h)$

2, $\mathrm{do} \circ u = id_{\mathcal{C}_1}$ & $u \circ id = id_{\mathcal{C}_2}$

in which $u : \mathcal{C}_0 \to \mathcal{C}_1$ is unit.

3, $f \square u(a) = f = u(b) \square f$ if

$\mathrm{do}(f) = a$ and $\mathrm{co}(f) = b$.

Notations: $\mathrm{Hom}_{\mathcal{C}}(a, b) :=$ Pull back of the following diagram :

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(a,b) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{C}_1 \arrow[d, "d_0 \times \mathcal{C}_0"] \\ \{(a,b)\} \arrow[r, hook] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

$\mathrm{Hom}(\mathcal{C}) := \bigsqcup_{a, b \in \mathcal{C}_0} \mathrm{Hom}_{\mathcal{C}}(a, b)$

Examples : 1, Category of Sets : $\mathcal{C}_0 = \{\text{Sets}\}$ $\mathcal{C}_1 = \{\text{map between sets}\}$

$\square =$ ordinary composition

2, Category of Hilbert space :

$\mathcal{C}_0 = \{\text{Hilbert spaces}\}$ $\mathcal{C}_1 = \{\text{unitary map}\}$

3, Category of Continuous map C_0: $\begin{matrix} E \\ \downarrow f \\ B \end{matrix}$ C_1

$$\begin{tikzcd} E \arrow[r, "G"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[d, "f'"] \\ B \arrow[r, "G'"'] & B' \end{tikzcd}$$

4, Fix a top space B define $\begin{matrix} B \\ \downdownarrows \\ B \end{matrix}$ $obj(\begin{matrix} B \\ \downdownarrows \\ B \end{matrix}) = B$, $Hom(\begin{matrix} B \\ \downdownarrows \\ B \end{matrix}) = B$

$d_0 = c_0 = u = id$, $\square(\begin{matrix} B \\ \downdownarrows \\ B \end{matrix}) = B \times_B B \cong B$

5, G a group, define $\begin{matrix} G \\ \downdownarrows \\ * \end{matrix}$

$obj(\begin{matrix} G \\ \downdownarrows \\ * \end{matrix}) = \{1\}$, $Hom(\begin{matrix} G \\ \downdownarrows \\ * \end{matrix}) = G$

$d_0 = c_0 =$ constant $\square :=$ group multiplication

6, $X =$ space, $G =$ group $\overset{\text{action}}{\cdot : X \times G \to X}$, define $\begin{matrix} X \times G \\ \downdownarrows \\ X \end{matrix}$

$obj(\begin{matrix} X \times G \\ \downdownarrows \\ X \end{matrix}) = X$, $Hom(\begin{matrix} X \times G \\ \downdownarrows \\ X \end{matrix}) = X \times G$

$$\begin{tikzcd} X & x.g \arrow[l, "{(x,g)}"'] & x.g.h \arrow[l, "{(x.g,h)}"'] \end{tikzcd}$$

7: Fix a top space B:

$obj(\begin{matrix} E \\ \text{Cont}\downarrow \\ B \end{matrix}, \alpha \in H^3(E, \mathbb{Z}))$ $mor(p\begin{matrix} E \\ \downarrow \\ B \end{matrix}, \alpha) \longrightarrow (\begin{matrix} E' \\ \downarrow p' \\ B \end{matrix}, \alpha')$

such that

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & \circlearrowright & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

, $\alpha = f^* \alpha' + p^* x$

Def: A groupoid is a category where all morphisms are isomorphism

Def: A topological groupoid is a groupoid such That all structure maps are continuous between l.cpt + T_2 + 2nd Cont. top spaces C_1, C_0.

Example 4, 5, 6 are top. groupoids.

7,

Def. Let $G_1 \rightrightarrows G_0^*$ be a top groupoid. A <u>Haar system</u> on $G_1 \rightrightarrows G_0$ is a family of (Borel) measures $\{\lambda^a\}_{a\in G_0}$ on G_1 such that

a) $\operatorname{supp} \lambda^a \subset d_0^{-1}(\{a\}) \subset G_1$

b) $(\gamma^*)_! \lambda^a = \lambda^b$ where $\gamma : b \to a$

$\gamma^* : d_0^{-1}(a) \to d_0^{-1}(b)$

$\varepsilon \mapsto \varepsilon \circ \gamma$

(In Term of integrals)

$$\int_{d_0^{-1}(\{b\})} F(\varepsilon\circ\gamma)\, d\lambda^b(\varepsilon) \overset{\text{Def}}{=} \int_{d_0^{-1}(\{a\})} F(\varepsilon)\, d\gamma^*_!\lambda^b(\varepsilon) \overset{(b)}{=} \int_{d_0^{-1}(\{a\})} F(\varepsilon)\, d\lambda^a(\varepsilon)$$

c) if $f \in C_c(G_1)$, then $G_0 \ni a \mapsto \int_{G_1} f\, d\lambda^a \in \mathbb{C}$ is Continuous

" (X,μ) = Borel measurable space, (a) $\operatorname{supp}\mu = X \setminus \bigcup_{\substack{u\subseteq X \\ \text{open} \\ \mu(u)=0}} u$, (b) $\varphi : X \to Y$ measurable map

$\varphi_!\mu$ = Push forward of μ along φ by

$\varphi_!\mu(A) := \mu(\varphi^{-1}(A))$.

Construct Haar systems in examples:

4) $B \rightrightarrows B$ $\quad d_0^{-1}(\{a\}) = \{a\}$ $\quad \lambda^a(\{A\}) = \begin{cases} 1 & a\in A \\ 0 & a\notin A \end{cases}$ $\quad A \subset G_1$

5) Special case of example 6.

6) $X \times G \rightrightarrows X$ $\quad x\cdot g^{-1} \xleftarrow{(xg^{-1},\, g)} x \xleftarrow{(x,\, h)} x\cdot h$

- $d_0^{-1}(x) = \{(x\cdot g^{-1}, g) : g\in G\} \xleftarrow[\cong]{\varphi_x} G$
- Let λ be a Haar meas. of G then $(h^*)_!\lambda = \lambda$ in which $h^* : G \to G$, $g \mapsto g\cdot h$

- Set $\lambda^{x} = \varphi_{x!}\lambda$

$$(x,h)^{*}_{!}\lambda^{x} = (x,h)^{*}_{!}\varphi_{x,!}\lambda = ((x,h)^{*}\circ\varphi_{x})_{!}\lambda = (\varphi_{xh}\circ h^{*})_{!}\lambda = \varphi_{xh!}h^{*}_{!}\lambda$$

$$\begin{tikzcd}
g \arrow[r, mapsto, "\varphi_x"] & (x.g^{-1}, g) \arrow[r] & (xg^{-1}, g.h) \arrow[d, leftrightarrow, bend left, "="] \\
g \arrow[r, mapsto, "h^{*}"] & gh \arrow[r, mapsto] & (xh(gh)^{-1}, gh)
\end{tikzcd}$$

$$= \varphi_{xh!}\lambda = \lambda^{xh}$$

Ansgar (II)

We introduce the top groupoids

$$\begin{tikzcd} B \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] & X \times G \arrow[d, shift left] \arrow[d, shift right] & B \times B \arrow[d, shift left] \arrow[d, shift right] \\ B & * & X & B \end{tikzcd}$$

- Haar system: family of Borel measure $\{\lambda^a\}_{a \in G_0}$, $\begin{tikzcd} G \arrow[d, shift left] \arrow[d, shift right] \\ G_0 \end{tikzcd}$ top gr

a) $\mathrm{supp}\, \lambda^a = d_0^{-1}(a)$

b) invariance

c) continuity

Ex: $\begin{tikzcd} B \times B \arrow[d, shift left] \arrow[d, shift right] \\ B \end{tikzcd}$ $d_0^{-1}(b) = \{(a,b) \in B \times B\}$

$$\begin{tikzcd} b^a & (a,b) \\ B \arrow[u, "\varphi_b"] & a \arrow[u, mapsto] \end{tikzcd}$$

Choose a measure μ on B with $\mathrm{supp}\, \mu = B$

$\lambda^b = \varphi_{b!} \mu$ $\quad (b,c)^*_! \lambda^b = \lambda^c$

$(b,c)^*_! \varphi_{b!} \mu = ((b,c)^* \varphi_b \ldots)_! \mu =$

$\varphi_c ! \mu = \lambda^c$

This exaple shows that the existence of Haar system is not unique.

Let $\begin{tikzcd} G \arrow[d, shift left] \arrow[d, shift right] \\ G_0 \end{tikzcd}$ be a top groupoid.

Fix $(\lambda^a)_{a \in G_0}$ some Haar system.

Consider $C_c(G_1)$ and define

$$* : C_c(G_1) \to C_c(G_1)$$
$$f \mapsto (g \mapsto \overline{f(g^{-1})})$$

- check $f^{**} = f$

$$* : C_c(G_1) \times C_c(G_1) \to C_c(G_1)$$
$$(f, f') \mapsto f * f'$$

$$f * f'(g) := \int f(\gamma \delta^{-1}) f'(\delta) \, d\lambda^{d_0(g)}(\delta)$$

claim: $*$ is associative

(convolution)

Proof:

$(f * f') * f''(\gamma) =$

$\int (f * f')(\gamma \delta^{-1}) f''(\delta^{-1}) \, d\lambda^{d_0(\gamma)}(\delta) =$

$\iint f(\gamma \delta^{-1} \epsilon) f'(\epsilon^{-1}) \, d\lambda^{d_0(\delta^{-1})} f''(\delta^{-1}) \, d\lambda^{d_0}(\delta)$

$= \ldots$:(

check $(f * f')^* = f'^* * f^*$

i.e

$(C_c(G), *, *)$ is an

involutive algebra.

Define a norm

$\|f\|_{\infty,1} = \underset{max}{\cancel{\sup}} \{\sup \|f\|_{L^1(G_1, \lambda^a)} ; a \in G_0\} \sup \|f^*\|_{L^1(G_1, \lambda^a)} ; a \in G_0\}$

- Get a Banach $*$-alg $\overline{C_c(G_1)}^{\|\cdot\|_{\infty,1}}$

LEMMA: This Banach $*$-alg has a (bd) approximate identity.

(This lemma enables us to define enveloping C^*-alg)

$* \quad C^*\left(\begin{smallmatrix} G_1 \\ \downdownarrows \\ G_0 \end{smallmatrix}\right)$ = enveloping C^*-alg of $\overline{C_c(G_1)}^{\|\ \|_{\infty,1}}$.

claim: $C^*\left(\begin{smallmatrix} B \\ \downdownarrows \\ B \end{smallmatrix}\right) = C_0(B)$

Pf: $(f * f')(a) = \int f(a b^{-1}) f'(b)\, d\lambda^{d_0(a)}(b) =$

$= \int f(a a^{-1}) f(a)\, d\lambda^{d_0(a)}(a) =$

$f(a) \cdot f'(a)$

$*$ = coplex convolution $\qquad f^* = \bar{f}$

$\|\ \|_{\infty,1} = \|\ \|_{\sup}$

claim $C^*\left(\begin{matrix}G\\ \downdownarrows\\ *\end{matrix}\right) = C^*(G)$,

claim $C^*\left(\begin{matrix}G\times X\\ \downdownarrows\\ X\end{matrix}\right) \cong C_0(X)\rtimes G$,

claim : $C^*\left(\begin{matrix}B\times B\\ \downdownarrows\\ B\end{matrix}\right) \cong k(L^2(B,\mu))$

Proof,
$$\begin{tikzcd} a & b \arrow[l, bend right, "{(a,b)}"'] \arrow[r, bend right, "{(c,b)}"'] & c \arrow[l, bend right, "{(b,c)=(c,b)^{-1}}"'] \end{tikzcd}$$

$$(f*f')(a,b) = \int f((a,b)\circ(c,b)^{-1})\, f'((c,b))\, d\lambda^{d\alpha(b)}((c,b)) \qquad \varphi_b(c)=(c,b)$$

$$= \int f((a,b)\circ\varphi_b(c)^{-1})\, f'(\varphi_b(c))\, d\mu(c)$$

$$= \int f(a,c)\, f'((c,b))\, d\mu(c)$$

Hilbet - Schmidt operators

$k_F : L^2(B,\mu) \to L^2(B,\mu)$

$k_f(F) = \int f(\cdot,b)\, F(b)\, d\mu(b)$

$\overline{HSop}^{\|\ \|op} = k_0(L^2(B,\mu))$

☺

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Goal: construct a 2-functor

$$\begin{tikzcd} Grpd \arrow[r] & C^*\text{-}algs \\ \text{2-groupoid of top groupoid} \arrow[u] & \text{2-groupoid of } C^*\text{-algebra} \arrow[u] \end{tikzcd}$$

DEF Let Z be a space. An action of a top groupoid $\left(\begin{matrix}G\\ \downdownarrows\\ G_0\end{matrix}\right)$ consists of

a) $Z \xrightarrow{q} G_0$

b) $\cdot : Z \times_{q,r} G_1 \longrightarrow Z$

c) $q(z\cdot\gamma) = d(\gamma)$

s.t

$(z\cdot\gamma)\cdot\delta = z\cdot(\gamma * \delta)$ if defined

$$\begin{tikzcd}
& Z \arrow[dl, "q'"'] \arrow[drr, "q"] & \circlearrowleft & G \arrow[d, shift left] \arrow[d, shift right] \\
Z/G_1 & & & G_0
\end{tikzcd}$$

(right action)

(quotient by the action)

* Analogously left actions

$$\begin{tikzcd}
G \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & Z \arrow[dll, "q"'] \arrow[dr, "q'"] & \\
G_0 & & & Z/G_1
\end{tikzcd}$$

The action G is called

- free if $z\cdot\gamma = z \Rightarrow \gamma$ is a unit

- proper if $$\begin{tikzcd}
Z \times G_1 \arrow[r, "(q{,}\omega)"'] & Z\times Z \\
(z,\gamma) \arrow[r, mapsto] & (z, z\cdot\gamma)
\end{tikzcd}$$

is a proper map.

<u>DEF</u>: A Morita equivalence between two top grpds $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$

is

$$\begin{tikzcd}
G_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & Z \arrow[dll, "p"'] \arrow[dl, two heads, "q'"'] \arrow[dr, "p'"] \arrow[drr, "q"] & \circlearrowleft & H_1 \arrow[d, shift left] \arrow[d, shift right] \\
G_0 & Z/H_1 & & Z/G_1 & H_0
\end{tikzcd}$$

homeomorph $\cong Z/H_1$ $\quad$ $Z/G_1 \cong$ homeomorph

s.t.

two actions, commute

- are free
- are proper

______ : Think of this as $Z : \begin{matrix} G_1 \\ \downdownarrows \\ G_1 \end{matrix} \longrightarrow \begin{matrix} H_1 \\ \downdownarrows \\ H_0 \end{matrix}$

Let
$$\begin{tikzcd} G_1 \arrow[r, "F_1"] \arrow[d, shift left] \arrow[d, shift right] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 \arrow[r, "F_0"'] & H_0 \end{tikzcd}$$
be a cts functor, then define

$$Z_F := G_0 \times_{F_0, c_0} H_1 = \{ (a, \gamma) : a \underset{F_0}{\mapsto} \cdot \overset{\gamma}{\curvearrowleft} \}$$

Z_F has a right H_1-action

$$Z_F \xrightarrow{q} H_0 \qquad\qquad Z_F \times_{q, c_0} H_1 \to Z_F$$

$$(z, \gamma) \longmapsto d_0(\gamma) \qquad\qquad ((z, \gamma), \delta) \mapsto (z, \gamma * \delta)$$

$$Z_F / H_1 \cong G_0$$

Z_F has canonical left action by G_1

$$Z_F \xrightarrow{p} G_0 \qquad\qquad G_1 \times_{d_0, p} Z_F \to Z_F$$

$$(a, \gamma) \longmapsto a \qquad\qquad (\alpha, (a, \gamma)) \mapsto (c_0(\gamma), F_1(\alpha) * \gamma)$$

Remark: Think of Z_F as a morphism $Z_F : \begin{matrix} G_1 \\ \downdownarrows \\ G_0 \end{matrix} \to \begin{matrix} H_1 \\ \downdownarrows \\ H_0 \end{matrix}$

Thm If (F_1, F_2) is an isomorphism, then Z_F is Morita equivalence.

Pf: $G_0 \times_{F_1, c_0} H_1 \overset{\psi}{\cong} G_1 \times_{d_0, F_0^{-1}} H_0$

Note that ψ is bi-equivariant for the canonical actions.

$$\begin{tikzcd} & y \arrow[d, "\gamma"'] & & \cdot \arrow[d, "F_1^{-1}(\gamma)"] & y \arrow[l, "F_0^{-1}"'] \\ a \arrow[r, mapsto, "F_0"'] & x \arrow[r, phantom, "\longmapsto"] & {} & a & \end{tikzcd}$$

Angar (III)

Recall:

$$\begin{tikzcd} & Z \arrow[loop left] \arrow[loop right] \arrow[dl] \arrow[dr] & \\ G_1 \arrow[d, shift left] \arrow[d, shift right] & & H_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & H_0 \end{tikzcd}$$

Morita equivalences are called 1-morphisms

$$C^*\left(\begin{smallmatrix} X/G \\ \downarrow\downarrow \\ X/G \end{smallmatrix}\right) \cong C_0(X/G) \xrightarrow{\overset{\text{Morita}}{\cong}} C^*\left(\begin{smallmatrix} X \times G \\ \downarrow\downarrow \\ X \end{smallmatrix}\right) \cong C_0(X) \rtimes G$$

Example: Let $X \times G \to X$ free and Proper action

$$\begin{tikzcd} & X \arrow[loop left] \arrow[loop right] \arrow[dl] \arrow[dr, "id"'] & \\ X/G \arrow[d, shift left] \arrow[d, shift right] & & X \times G \arrow[d, shift left] \arrow[d, shift right] \\ X/G & & X \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[d, mapsto] & \\ x & x.g \arrow[l, "(x,g)"'] \end{tikzcd}$$

Non-example: Consider $S^1 \times \mathbb{Z} \to S^1$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$

$$(z, n) \mapsto z e^{2\pi i n \theta}$$

this action is free but not proper.

The quotient $S^1/\mathbb{Z}$ has trivial topology:

$$\tau = \{\phi, S^1/\mathbb{Z}\}.$$

So $C(S^1/\mathbb{Z}) \cong \mathbb{C}$

Starting Point of non-Comm. Topology

$$\text{Category of cpt} + T_2 \text{ Space} \xrightarrow{\sim} \left(\begin{smallmatrix} \text{unital} + \text{Comm} \\ C^*\text{-alg} \end{smallmatrix}\right)^{op}$$

$$\subseteq (\text{unital } C^*\text{-alg})^{op}$$

!!

$C^*\left(\begin{smallmatrix} S^1 \times \mathbb{Z} \\ \downarrow\downarrow \\ S^1 \end{smallmatrix}\right) \in$ Cat of non-comm. cpt top space

How to compose 1-morphisms?

$$\begin{tikzcd} G_1 \arrow[d, shift left] \arrow[d, shift right] & Y \arrow[dl, "p"] \arrow[dr, "q"'] & H_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[dl, "r"] \arrow[dr, "s"'] & K_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & H_0 & & K_0 \end{tikzcd}$$

$$(y, z).\gamma = (y.\gamma, \gamma^{-1}.z)$$

Define $Y \square Z := Y \times_{H_0} Z / H_1$

$$\begin{tikzcd} {[\gamma.y, z]} \arrow[d, mapsto] & & & & {[y, z.\delta]} \arrow[d, mapsto] \\ \cdot & & {[y, z]} \arrow[dl, mapsto] \arrow[dr, mapsto] & & \cdot \arrow[dl, "\delta"] \\ & p(y) \arrow[ul, bend left, "\gamma"] & & q(z) & \end{tikzcd}$$

These actions Commute and proper.

What about associativity?

$$(Y \square Z) \square W \neq Y \square (Z \square W)$$

$$\cong (\text{biequivariant})$$

$$[[y, z]_H, w]_K \longrightarrow [y, [z, w]_K]_H$$

Def: A 2-morphisms is bi-equivar. homeomorphism:

$$\begin{tikzcd} & Z' \arrow[ddl] \arrow[ddr] & \\ G_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[u, "\cong"] \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & H_0 \end{tikzcd}$$

Observe : • 2-morphisms can be composed

vertically

$$\begin{tikzcd}
G_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[r, "\cong", "X"'] \arrow[dl] \arrow[drrr] & Z' \arrow[r, "\cong", "X'"'] \arrow[dll] \arrow[drr] & Z'' \arrow[dlll] \arrow[dr] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\
G_0 & & & & H_0
\end{tikzcd}$$

• 2-morphisms can be composed

Horizentally

$$\begin{tikzcd}
Z \arrow[r, "\cong", "X"'] \arrow[d] \arrow[dr] & Z' \arrow[dl] \arrow[d] & W \arrow[r, "\cong", "Y"'] \arrow[dl] \arrow[dr] & W' \arrow[dll] \arrow[d] \\
G_0 & H_0 & & K_0
\end{tikzcd}$$

$X \Delta Y : Z \square W \cong Z' \square W'$

• $(X \circ X') \Delta (Y \circ Y') = (X \Delta Y) \circ (X' \Delta Y')$

Note: associativity satisfies the

Pentagon relation:

$$\begin{tikzcd}
 & ((W \square X) \square Y) \square Z \arrow[dl, "ass(W,X,Y) \Delta id_Z"'] \arrow[dr, "ass."] & \\
(W \square (X \square Y)) \square Z \arrow[d, "ass."'] \arrow[r, phantom, "\circlearrowright"] & & (W \square X) \square (Y \square Z) \arrow[d, "ass."] \\
W \square ((X \square Y) \square Z) \arrow[rr, "id_W \Delta ass."'] & & W \square (X \square (Y \square Z))
\end{tikzcd}$$

What about units?

$$\begin{tikzcd}
G_1 \arrow[d, shift left] \arrow[d, shift right] & G_1 \arrow[dl, "c_0"] \arrow[dr, "d_0"] & G_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\
G_0 & & G_0 & & H_0
\end{tikzcd}$$

$l : G_1 \square Z \underset{\cong}{\neq} Z$

$r : Y \square G_1 \underset{\cong}{\neq} Y$

$$\begin{tikzcd}
G_1 \square G_1 \arrow[r, "l"] \arrow[d, "r"'] & G_1 \arrow[dl, equal, "id"] \\
G_1 &
\end{tikzcd}$$

Def: A 2-groupoid (weak 2 groupoid)

consists of
- set of objects
- " " 1-morphisms
- " " 2-morphisms
- $\square, \circ, \Delta, l, r, i, ass, ^{-1}$.

such that various coherence conditions are satisfied.

observation: C^*-alg form a 2-groupoid

- ob = C^*-alg
- 1-morphism = Morita equivalence
- 2-morphism = Iso. of Morita equiv.

Ansgar (IV)

Let
$$\begin{tikzcd} G_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[dl, "p"'] \arrow[dr, "q"] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & H_0 \end{tikzcd}$$

be a morita equivalence.

Let $\{\lambda^a\}_{a \in G_0}$ and $\{\rho^a\}_{a \in H_0}$ be

Haar systems on G_0, H_0.

Take $\varphi, \psi \in C_c(Z)$. for $f \in C_c(G_1)$, $g \in C_c(H_1)$

we define for any $z \in Z$

i $(f \cdot \varphi)(z) = \int f(\varepsilon^{-1}) \varphi(\varepsilon . z)\, d\lambda^{p(z)}(\varepsilon)$

ii) $(\varphi \cdot g)(z) = \int \varphi(z . \gamma^{-1}) g(\gamma)\, d\rho^{q(z)}(\gamma)$

iii) $\langle \varphi, \psi \rangle_H (\eta) = \int \overline{\varphi(\gamma . z)}\, \psi(\gamma z . \eta)\, d\lambda^{p(z)}(\gamma)$

iv) ${}_G\langle \varphi, \psi \rangle (\gamma) = \int \varphi(\gamma z \eta^{-1}) \overline{\psi}(z . \eta^{-1})\, d\rho^{q(z)}(\eta)$

Claim: The functions defined in (i)-(iv) have cpt support. and

a) $f \cdot (f' \cdot \varphi) = (f *_{\lambda} f') \cdot_{e} \varphi$

b) $(\varphi \cdot g) \cdot g' = \varphi \cdot (g *_{\rho} g')$

c) $\langle \varphi, \psi \cdot g \rangle_H = \langle \varphi, \psi \rangle_H \cdot g$

d) ${}_G\langle f \cdot \varphi, \psi \rangle = f \cdot {}_G\langle \varphi, \psi \rangle$

e) $\langle \varphi_1, \varphi_2 \rangle_H^* = \langle \varphi_2, \varphi_1 \rangle_H$

f) ${}_G\langle \varphi_1, \varphi_2 \rangle^* = {}_G\langle \varphi_2, \varphi_1 \rangle$

g) ${}_G\langle \varphi \cdot g, \psi \rangle = {}_G\langle \varphi, \psi \cdot g^* \rangle$

h) $\langle \varphi, f \cdot \psi \rangle_H = \langle f^* \cdot \varphi, \psi \rangle_H$

i) ${}_G\langle \varphi, \psi \rangle \cdot \chi = \varphi \cdot \langle \psi, \chi \rangle_H$

Check: Density and Continuity

$$\|\varphi\|_Z := \|\langle \varphi, \varphi \rangle\|^{1/2}_{C^*\left(\begin{smallmatrix} G_1 \\ \downdownarrows \\ G_2 \end{smallmatrix}\right)} = \|\langle \varphi, \varphi \rangle_H\|^{1/2}_{C^*\left(\begin{smallmatrix} H_1 \\ \downdownarrows \\ H_0 \end{smallmatrix}\right)}$$

Conclusion: getting a Morita equivalence

$$\overline{C_c(Z)}^{\|\cdot\|_Z}$$

$$C^*\left(\begin{smallmatrix} G_1 \\ \downdownarrows \\ G_0 \end{smallmatrix}\right) \qquad C^*\left(\begin{smallmatrix} H_1 \\ \downdownarrows \\ H_0 \end{smallmatrix}\right)$$

stat with

$$\begin{tikzcd} G_1 \arrow[d, shift left] \arrow[d, shift right] & Y \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[dl] \arrow[dr] & K_1 \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & H_0 & & K_0 \end{tikzcd}$$

Then

$$\overline{C_c(Y)}^{\|\cdot\|_Y} \qquad \overline{C_c(Z)}^{\|\cdot\|_Z}$$

$$C^*\left(\begin{smallmatrix} G_1 \\ \downdownarrows \\ G_0 \end{smallmatrix}\right) \qquad C^*\left(\begin{smallmatrix} H_1 \\ \downdownarrows \\ H_0 \end{smallmatrix}\right) \qquad C^*\left(\begin{smallmatrix} K_1 \\ \downdownarrows \\ K_0 \end{smallmatrix}\right)$$

$$:= \overline{C_c(\cancel{Y})}^{\|\cdot\|} \boxtimes_{C^*\left(\begin{smallmatrix} H_1 \\ \downdownarrows \\ H_0 \end{smallmatrix}\right)} \overline{C_c(Z)}^{\|\cdot\|}$$

$$\cong \overline{C_c(Y \square Z)}^{\|\cdot\|_{Y \square Z}}$$

Start with :

$$\begin{tikzcd}
G_1 \arrow[d, shift left] \arrow[d, shift right] & Z \overset{\sim}{=} Z' \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift left] \arrow[d, shift right] \\
G_0 & & H_0
\end{tikzcd}$$

Then we get

$$\overline{C_c(Z)}^{\|\cdot\|} \overset{\sim}{=} \overline{C_c(Z')}^{\|\cdot\|}$$

equivariant

Conclusion : We have a 2-functor :

$$\text{Groupoid} \longrightarrow C^*\text{-alg}$$

Top. groupoids are a starting point for the study of top. stacks :

Fix X a space. Consider the category of principal $\left(\begin{smallmatrix} G_1 \\ \downarrow\downarrow \\ G_0 \end{smallmatrix}\right)$ - bundles

obj

$$\begin{tikzcd}
E \arrow[loop above] \arrow[d, "p"'] \arrow[dr, "q"] & G_1 \arrow[d, shift left] \arrow[d, shift right] \\
X & G_0
\end{tikzcd}
\qquad E \times_{G_0} G \xrightarrow{\overset{\sim}{=}} E \times_{G_0} E$$

has local sections

mor :

$$\begin{tikzcd}
E \arrow[rr, "\text{equivariant}"] \arrow[dr] & & E' \arrow[dl] \\
& X &
\end{tikzcd}$$

This category is groupoid and its denoted by $BG(X)$.

Let $Top(X)$ be the category of open sets of

$obj = \{U \subseteq X : \text{open}\}$ $\quad mor = \left\{ \begin{smallmatrix} U \subset U' \\ \cap\ \ \cap \\ X \end{smallmatrix} \right\}$

Then the assignment

$$BG : Top(X) \longrightarrow \text{Set theoretical groupoids}$$

$$U \longmapsto BG(U)$$

is a top. stack.

Infact

$$\text{Groupoid} \xrightarrow{\sim} \text{top stacks over } X$$

$$\begin{matrix} G_1 \\ \downarrow\downarrow \\ G_0 \end{matrix} \longmapsto BG(\cdot)$$